PROBLEM SET 2

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Exercise 1. Let $K = \mathbb{Q}(\sqrt{n})$ with n square free. If $n \equiv 1 \mod 4$, then $\mathcal{O}_K = \mathbb{Z}[\frac{1+\sqrt{n}}{2}]$ and $d_K = n$. If $n \equiv 2, 3 \mod 4$, then $\mathcal{O}_K = \mathbb{Z}[\sqrt{n}]$ and $d_K = 4n$. Concisely written we have $\mathcal{O}_K = \mathbb{Z}[\frac{d_K+\sqrt{d_K}}{2}]$ in any case.

Exercise 2. The general class number formula yields the following way to compute h_K for an imaginary quadratic field $K = \mathbb{Q}(\sqrt{n}), n < 0$ square free. Let w_K be the number of roots of unity in K. (4 for $\mathbb{Q}(i)$, 6 for $\mathbb{Q}(\sqrt{-3})$, 2 otherwise.) Let $N = |d_K|$. Then

$$h_K = -\frac{w_K}{2N}\sum_{a=1}^{N} a\chi(a),$$

where $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \{\pm 1\}$ is a homomorphism characterized by the condition $\chi(p) = (\frac{n}{p})$ for odd primes p coprime to n. Note that although the existence of such a χ is nontrivial, we can compute the values of χ using the characterization. Compute the class numbers of $\mathbb{Q}(\sqrt{-5}), \mathbb{Q}(\sqrt{-6}), \mathbb{Q}(\sqrt{-10})$.

Exercise 3. Let K be a quadratic extension of $\mathbb{Q}$ with discriminant d. Prove the following: Let p be an odd prime. We have

(1) p is ramified in K if and only if $p|d$. We have $(p) = (p, \sqrt{d/4})^2$ when $4|d$, and $(p) = (p, \theta)^2$ when $4 \nmid d$.
(2) Let p be prime to d and suppose $(\frac{d}{p}) = 1$. Then p is split. If $4|d$, we have $(p) = (p, \sqrt{d/4} - a)(p, \sqrt{d/4} + a)$ where $a \in \mathbb{Z}$ is any solution to $a^2 \equiv d/4 \mod p$. If $4 \nmid d$, we have $(p) = (p, \theta - (d+a)b)(p, \theta - (d-a)b)$, where $a, b \in \mathbb{Z}$ are any solutions to $a^2 \equiv d, 2b \equiv 1 \mod p$.
(3) If p is prime to d and $(\frac{d}{p}) = -1$, then p is inert in K.

Moreover, 2 is ramified in K if and only if $4|d$, in which case $(2) = (2, \sqrt{d/4} - d/4)^2$. Suppose $4 \nmid d$. When $\frac{d-1}{4}$ is odd, 2 is inert. When $\frac{d-1}{4}$ is even, $(2) = (2, \theta)(2, \theta + 1)$ is split.